

Today, using some examples, let's look at different ways of distributing identical/distinct objects among people or in groups. There are some formulas which can be used in some of these cases but I will only discuss how to use the concepts we have learned so far to deal with these questions. I am not a fan of unintuitive formulas since the probability (we will come to this topic soon) that we will get to use even one of them in GMAT is quite low while the effort involved in cramming all of them is humongous. Therefore, I only want to focus on our core concepts which we can apply in various situations. Let's start with our first example.

Question 1: In how many ways can 5 different fruits be distributed among four children? (Some children may get more than one fruit and some may get no fruits.)

- (A)  $4^5$
- (B)  $5^4$
- (C)  $5!$
- (D)  $4!$
- (E)  $4! \cdot 5!$

Solution 1: This is the simplest case – 5 different things are to be distributed among 4 different people. Say, the five different fruits are: an apple, a banana, a strawberry, an orange and a blueberry. In how many ways can you give an apple to someone? In 4 ways. You can give the apple to any one of the 4 children. Similarly, in how many ways can you give a banana to someone? Again in 4 ways. It is acceptable to give more than one fruit to the same child so you again have 4 possibilities for the banana. Similarly, all the remaining fruits can also be distributed in 4 ways each.

Total number of ways of distributing 5 different fruits among 4 children =  $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$  ways.

Answer (A)

Another popular variation on this type of question is this:

Question 1a: In how many ways can 5 different rings be worn in four particular fingers? (Some fingers may get more than one ring and some may get no rings.)

Here, the answer will be a little different. Think why.

Question 2: In how many ways can 5 apples (identical) be distributed among 4 children? (Some children may get no apples.)

- (A) 56
- (B) 144
- (C) 200
- (D) 256
- (E) 312

Solution: We have 5 identical apples and 4 children. We want to find the number of ways in which these apples can be distributed among the children.

#### Method I

5 apples can be distributed in various ways:  $\{5, 0, 0, 0\}$ ,  $\{4, 1, 0, 0\}$ ,  $\{3, 2, 0, 0\}$ ,  $\{3, 1, 1, 0\}$ ,  $\{2, 2, 1, 0\}$ ,  $\{2, 1, 1, 1\}$ .

$\{5, 0, 0, 0\}$  means that one child gets all 5 apples and all others get none. Similarly,  $\{4, 1, 0, 0\}$  means that one child gets 4 apples and another child gets 1 apple. No one else gets any apples and so on...

In each one of these cases, various arrangements are possible e.g. take the case of {5, 0, 0, 0}. The first child could get all 5 apples OR the second child could get all 5 apples OR the third child could get all 5 apples OR the fourth child could get all 5 apples. Basically, the number of ways in which these 4 objects – 5, 0, 0 and 0 can be distributed in 4 different spots (i.e. 4 children) is  $4!/3! = 4$  arrangements (we divide by  $3!$  because three of the objects – 0, 0 and 0 – are identical). This is just our beloved basic counting principle in action.

Similarly, in the case of {4, 1, 0, 0}, we will get  $4!/2! = 12$  arrangements (since 2 objects are identical) i.e. 5 apples can be distributed among 4 children by giving 4 apples to one child and 1 apple to another child in 12 ways. The first child could get 4 apples and the second child could get 1 apple OR the third child could get 4 apples and the first child could get 1 apple etc.

In the same way, we will get 12 arrangements in each one of these cases: {3, 2, 0, 0}, {3, 1, 1, 0} and {2, 2, 1, 0}. In the case of {2, 1, 1, 1}, we will get  $4!/3! = 4$  arrangements.

In all, 5 identical apples can be distributed among 4 children in  $4 + 12 + 12 + 12 + 12 + 4 = 56$  ways

Here, we have just counted out the ways in which 5 things can be distributed in 4 groups. If we miss even one of these cases, all our effort would go waste. Therefore, let's look at a more analytical method of solving this question.

## Method II

Let's put the 5 apples in a row: A A A A A

We have to split them in 4 groups. The 4 groups will have a one-to-one relation with the 4 children – Apples in the first group will be given to the first child, those in the second group will be given to the second child and so on...

Say we split the apples in 4 groups in the following way: A A | A | A | A

The vertical lines separate one group from the other. The first group has 2 apples and the rest of the three groups have 1 apple each. This means, the first child gets 2 apples and each of the other 3 children get 1 apple each.

The split can also be made in the following way: A | A A | A | A

Here, the second group has 2 apples and the rest of the three groups have 1 apple each. So the second child gets 2 apples and the rest of the 3 children get 1 apple each.

The split can also be made in the following way: | A A A | A | A

Here, the first group has no apples, the second group has 3 apples and the third and the fourth groups have one apple each. The first child gets no apples, the second child gets 3 apples and the other 2 children get 1 apple each.

What I am trying to show here is that arranging these 5 identical As and the 3 identical vertical lines in as many different ways as possible will give us all the ways in which we can distribute 5 identical apples among 4 different children.

In how many different ways can we arrange these 8 objects i.e. 5 identical As and 3 identical vertical lines? In  $8!/(5! * 3!) = 56$  ways

Answer (A).

We have seen how to solve this question in two different ways. A point to note here is that method II cannot be used in

question 1 above. Think why.

Now try to solve these two questions:

Question 3: In how many ways can 5 different fruits be distributed among 4 identical baskets?

Question 4: In how many ways can 5 apples (identical) be distributed among 4 identical baskets?

We will look at their solutions and the solution of the variation question next week!